



Hölder continuity of singular parabolic equations with variable nonlinearity

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Abstract

In this paper we obtain the local Hölder regularity of the weak solutions for singular parabolic equations with variable exponents. The proof is based on DiBenedetto's technique called intrinsic scaling; by choosing an appropriate geometry one can deduce energy and logarithmic estimates from which one can implement an iterative method to obtain the regularity result.

1 Introduction

Consider the following parabolic equations

$$u_t - \operatorname{div}A(x, t, u, \nabla u) = B(x, t, u, \nabla u) \text{ in } \Omega_T, \quad (1.1)$$

where $\Omega_T \equiv \Omega \times (0, T]$, Ω is a bounded simple-connected domain in \mathbb{R}^N and $0 < T < +\infty$. The functions $A : \Omega_T \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$ and $B : \Omega_T \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ are assumed to be measurable and satisfying the following structure conditions

$$|A(x, t, u, \nabla u)| \leq C_1 \left(\phi(x, t) + |u|^{p(x,t)-1} + |\nabla u|^{p(x,t)-1} \right), \quad (1.2)$$

$$|B(x, t, u, \nabla u)| \leq C_2 \left(\phi(x, t) + |u|^{p(x,t)-1} + |\nabla u|^{p(x,t)-1} \right), \quad (1.3)$$

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$$A(x, t, u, \nabla u) \cdot \nabla u \geq C_3 |\nabla u|^{p(x,t)}, \quad (1.4)$$

where $\phi(x, t) \in L^\infty(\Omega_T)$ and C_1, C_2, C_3 are positive constant. Throughout the paper we assume that the exponent $p(x, t)$ is a given measurable function in Ω_T such that

$$p(x, t) \in [p^-, p^+] \subset (1, 2).$$

Moreover, we assume that the exponent p satisfies the following log-continuity condition:

$$\begin{aligned} |p(x, t) - p(y, \tau)| &\leq \frac{c_0}{\ln \frac{1}{|t-\tau|+|x-y|}} \quad \text{for any } (x, t), (y, \tau) \in \Omega_T, \\ |t - \tau| + |x - y| &\leq \frac{1}{2}. \end{aligned} \quad (1.5)$$

The conditions like (1.2)-(1.4) are called $p(x, t)$ -growth conditions, which are special cases of nonstandard growth conditions. The parabolic equations with nonstandard growth conditions are the mathematical models of flows of electrorheological or thermorheological fluids, processes of filtration through a porous medium and the processing of digital images [1, 2,9,24]. Further applications of these parabolic equations can be seen in [4,5,13,20,21] or the references therein.

The operator $-\operatorname{div}A(x, t, u, \nabla u)$ is a generalization of the $p(x, t)$ -Laplace operator $-\operatorname{div}(|\nabla u|^{p(x,t)-2}\nabla u)$ and the generalized mean curvature operator $-\operatorname{div}((1 + |\nabla u|^2)^{\frac{p(x,t)-2}{2}}\nabla u)$ (see [19] for more details). Therefore, the problem (1.1) can be viewed as a generalization of the parabolic $p(x, t)$ -Laplace problem

$$u_t - \operatorname{div}(|\nabla u|^{p(x,t)-2}\nabla u) = B(x, t, u, \nabla u) \text{ in } \Omega_T,$$

and the generalized parabolic mean curvature problem

$$u_t - \operatorname{div}\left((1 + |\nabla u|^2)^{\frac{p(x,t)-2}{2}}\nabla u\right) = B(x, t, u, \nabla u) \text{ in } \Omega_T.$$

For more examples of problems that can be generalized by equation (1.1) see the monograph [4].

Our aim here is to obtain a local regularity result for local weak solutions of (1.1). In order to achieve this goal, and since the equation is singular (the modulus of ellipticity becomes unbounded at points where $|\nabla u| = 0$), the idea is to study the equation within a geometry that takes this feature into consideration. The building blocks of DiBenedetto's intrinsic scaling method is to show that the continuity of the solution at a point follows from measuring its oscillation in a sequence of nested and shrinking cylinders, with vertex at that point, and showing that the oscillation converge to zero as the cylinders

shrink to the point. To fully understand the technical procedure, based on the study of an alternative argument which makes use of energy and logarithmic estimates, one has not only to be familiar with DiBenedetto's technique (see [10,11,22]) but also to overcome the difficulty of having an (x, t) -dependence on the exponent p .

The development of the regularity theory for degenerate and/or singular parabolic pdes is one example of the contemporary analysis of nonlinear pdes. One has to go back to the final fifties to encounter the now standard procedure that allows one to get a regularity result for the solutions of nonlinear pdes: regularity for elliptic pdes was established by De Giorgi [12]; while Moser [14, 15, 16, 17], Nash [18] and DiBenedetto [10] dealt with parabolic pdes. Thorough analysis has been carried out for the case of the parabolic isotropic and anisotropic $p(x, t)$ – laplacian, in which Hölder continuity and other properties of solutions are established in [3, 7, 8].

2 preliminary and main results

2.1 Mollification in time

Since weak solutions of parabolic equations, respectively inequalities possess only weak regularity properties with respect to time, it is in principle not possible to use the solution itself as a test-function in the weak formulation of the problem. In order to be nevertheless able to test properly, there are several possibilities to smooth the solution with respect to the time direction. To overcome these faculties, we consider the Friedrichs mollifier as was done in [3]. Indeed, taking the kernel

$$\rho \geq 0, \quad \rho \in C_0^\infty(\mathbb{R}^N), \quad \rho(x) \equiv 0 \quad \text{for } |x| \geq 1, \quad \int_{\mathbb{R}^N} \rho(x) dx = 1,$$

we introduce regularization of $f \in L_{loc}^{p(x,t)}(\Omega_T)$ by

$$\begin{aligned} I^h f = f_h(x, t) &= h^{-1} \int_t^{t+h} \int_{|x-y| \leq h} f(y, \tau) \rho_h(x-y) dy d\tau, \\ \rho_h(x) &= h^{-N} \rho(h^{-1}x), \end{aligned} \quad (2.1)$$

and consider these inside the cylinder Ω_T , i.e., in cylinders $\Omega'_T = \Omega' \times (T_1, T_2)$, where $\Omega' \subset \Omega$, $0 < T_1 < T_2 < T$. The basic property of the mollification (2.1), which can be retrieved from [[3], Lemma 2.1], is summarized in the following:

Lemma 2.1. *If the exponent p satisfies the condition (1.5), then $f_h \rightarrow f$ in $L_{loc}^{p(x,t)}(\Omega_T)$ as $h \rightarrow 0$, for any $f \in L_{loc}^{p(x,t)}(\Omega_T)$.*

2.2 Formulation of the problem

We will consider here local weak solutions of equation (1.1), the existence of such solutions is guaranteed by [23,25].

Definition 2.2. *A local weak solution of (1.1) is a measurable function $u(x, t)$ defined in Ω_T such that*

$$(i) \quad u \in C(0, T; L^2(\Omega)) \cap L^{p^-}(0, T; W^{1,p(x,t)}(\Omega)),$$

(ii) *for every subset K of Ω and for every subinterval $[t_1, t_2]$ of $(0, T]$*

$$\begin{aligned} \left[\int_K u \phi \, dx \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \{-u \phi_t + A(x, t, u, \nabla u) \cdot \nabla \phi\} \, dx dt \\ = \int_{t_1}^{t_2} \int_K B(x, t, u, \nabla u) \phi \, dx dt, \end{aligned} \quad (2.2)$$

for all locally bounded tested functions

$$\phi \in W_{loc}^{1,2}(0, T; L^2(K)) \cap L_{loc}^{p^-}(0, T; W_0^{1,p(x,t)}(K)).$$

We can write (ii) in a way that is technically more convenient and involves the discrete time derivative. This can be accomplished by using the Friedrichs mollifier of a function (see [3] for more details). Then, we get the following:

Lemma 2.3. *If u is a solution of equation (1.1) in the sense of Definition 2.2, then for every subset K of Ω , and for any $h < t_1 \leq t_2 < T - h$, the following relation*

$$\int_{t_1}^{t_2} \int_K \{u_{h,t} \varphi + [A(x, t, u, \nabla u)]_h \cdot \nabla \varphi - [B(x, t, u, \nabla u)]_h \varphi\} \, dx dt = 0, \quad (2.3)$$

holds for any locally bounded tested function

$$\varphi \in W_{loc}^{1,2}(0, T; L^2(K)) \cap L_{loc}^{p^-}(0, T; W_0^{1,p(x,t)}(K)).$$

Proof. We introduce the following regularization operator:

$$I^{-h} f = f_{-h}(x, t) = h^{-1} \int_{t-h}^t \int_{|x-y| \leq h} f(y, \tau) \rho_h(x-y) \, dy d\tau. \quad (2.4)$$

Consider equation (2.2) with

$$\phi = I^{-h}(\varphi \chi), \quad \varphi \in W_{loc}^{1,2}(0, T; L^2(K)) \cap L_{loc}^{p^-}(0, T; W_0^{1,p(x,t)}(K)).$$

Since

$$-\int_{t_1}^{t_2} \int_K u \frac{\partial I^{-h}(\varphi\chi)}{\partial t} dxdt = \int_{t_1}^{t_2} \int_K u_{h,t} \varphi\chi dxdt,$$

it follows that

$$\int_{t_1}^{t_2} \int_K [u_{h,t} \varphi\chi + [A(x, t, u, \nabla u)]_h \cdot \nabla(\varphi\chi) - [B(x, t, u, \nabla u)]_h \varphi\chi] dxdt = 0.$$

Passing here from $\chi \in C_0^\infty(t_1, t_2)$ to characteristic function of the segment $[t_1, t_2]$, we obtain the desired relation (2.3). \square

The proof of the oscillation lemma, which implies the Hölder continuity of solutions $u(x, t)$ of equation (1.1), is based on integral estimates for truncated functions of the form

$$(u - k)_+ = \max(u - k, 0), \quad (u - k)_- = \max(-(u - k), 0), \quad k \in \mathbb{R}.$$

One of the required inequalities is given in the following lemma.

Lemma 2.4. *If u is a solution of equation (1.1) in the sense of Definition 2.2 and $v = (u - k)_\pm$, then for any $h < t_1 \leq t_2 < T - h$, the following relation*

$$\int_{t_1}^{t_2} \int_\Omega \{v_{h,t} \varphi + [A(x, t, v, \nabla v)]_h \cdot \nabla \varphi - [B(x, t, v, \nabla v)]_h \varphi\} dxdt \leq 0, \quad (2.5)$$

holds for any nonnegative tested function

$$\varphi \in W_{loc}^{1,2}(0, T; L^2(K)) \cap L_{loc}^p(0, T; W_0^{1,p(x,t)}(K)).$$

Proof. Taking

$$\varphi = \frac{(u_h - k)_\pm \phi}{(u_h - k)_\pm + \varepsilon}, \quad \phi \geq 0,$$

where

$$\frac{\partial \phi}{\partial t} \in L^2(\Omega_T), \quad \text{and } \phi = 0 \text{ in a neighborhood of } \partial\Omega_T,$$

as a test function in (2.2). Integrating by parts in the first term, using (2.1), and passing to the limit $h \rightarrow 0$ (this is possible due to the interpolation Lemma 2.1) and for $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned} & \int_{t_1}^{t_2} \int_\Omega \left\{ -v \frac{\partial \phi}{\partial t} + A(x, t, v, \nabla v) \cdot \nabla \phi - B(x, t, v, \nabla v) \nabla \phi \right\} dxdt \\ & \leq -\liminf_{\varepsilon \searrow 0} C\varepsilon \int_{t_1}^{t_2} \int_\Omega |\nabla v|^{p(x)} (v + \varepsilon)^{-2} \phi dxdt. \end{aligned}$$

Since the limit is nonnegative and t_1, t_2 are arbitrary, it follows that

$$\int_{t_1}^{t_2} \int_{\Omega} \left\{ -v \frac{\partial \phi}{\partial t} + A(x, t, v, \nabla v) \cdot \nabla \phi - B(x, t, v, \nabla v) \nabla \phi \right\} dx dt \leq 0.$$

Now, repeating the reasoning in the proof of Lemma 2.1, we come to the desired inequality (2.5). \square

2.3 Regularity result

In order to obtain the interior continuity of the solutions by means of intrinsic scaling, we need to consider a geometry that accommodates the singularity of the parabolic equation (1.1). For this purpose, fix a point $(x_0, t_0) \in \Omega_T$, assume $(x_0, t_0) = (0, 0)$. Consider a cylinder

$$Q(R^{p^+}, R^{\frac{p^+}{2}}) = K_{R^{\frac{p^+}{2}}} \times (-R^{p^+}, 0) := \{x : \max_{1 \leq i \leq N} |x_i| < R^{\frac{p^+}{2}}\} \times (-R^{p^+}, 0)$$

where $R > 0$ is taken such that the inclusion holds. Now, let

$$\mu^+ = \operatorname{ess\,sup}_{Q(R^{p^+}, R^{\frac{p^+}{2}})} u, \quad \mu^- = \operatorname{ess\,inf}_{Q(R^{p^+}, R^{\frac{p^+}{2}})} u, \quad \text{and} \quad \omega = \operatorname{ess\,osc}_{Q(R^{p^+}, R^{\frac{p^+}{2}})} u = \mu^+ - \mu^-,$$

and construct the cylinder

$$Q(R^{p^+}, c_0 R) \text{ with } c_0 = \left(\frac{\omega}{2^\lambda}\right)^{\frac{p^- - 2}{p^+}}$$

where λ is to determined only in terms of the data.

In order to study our problem inside $Q(R^{p^+}, R^{\frac{p^+}{2}})$, we start the iteration by assuming that

$$\left(\frac{\omega}{2^\lambda}\right)^{\frac{p^- - 2}{p^+}} \leq R^{\frac{p^+ - 2}{2}}. \quad (2.6)$$

Then, we obtain that

$$Q(R^{p^+}, c_0 R) \subset Q(R^{p^+}, R^{\frac{p^+}{2}}), \text{ and } \operatorname{ess\,osc}_{Q(R^{p^+}, c_0 R)} u \leq \omega.$$

Remark 2.5. *If (2.6) does not hold, then the essential oscillation ω goes to zero when the radius R goes to zero, and then there is nothing to prove.*

In order to begin our approach, Inside $Q(R^{p^+}, c_0 R)$ consider subcylinders of smaller size as follows:

$$(\bar{x}, 0) + Q(R^{p^+}, d_0 R), \quad d_0 = \left(\frac{\omega}{2}\right)^{\frac{p^- - 2}{p^+}}. \quad (2.7)$$

These cylinders are contained in $Q(R^{p^+}, c_0 R)$ if \bar{x} ranges over the cube $K_{R(\omega)}$ where

$$R(\omega) = \left\{ (2^{\lambda-1})^{\frac{p^- - 2}{p^+}} - 1 \right\} \left(\frac{\omega}{2} \right)^{\frac{p^- - 2}{p^+}} R = L_0 d_0 R, \quad (2.8)$$

for $L_0 = (2^{\lambda-1})^{\frac{p^- - 2}{p^+}} - 1$.

Now, given $\nu_0 \in (0, 1)$, to be determined in terms of the data, either

$$\left| \left\{ (x, t) \in (\bar{x}, 0) + Q(R^{p^+}, d_0 R) : u(x, t) < \mu^- + \frac{\omega}{2} \right\} \right| \leq \nu_0 \left| Q(R^{p^+}, d_0 R) \right|, \quad (2.9)$$

or

$$\left| \left\{ (x, t) \in (\bar{x}, 0) + Q(R^{p^+}, d_0 R) : u(x, t) < \mu^- + \frac{\omega}{2} \right\} \right| > \nu_0 \left| Q(R^{p^+}, d_0 R) \right|. \quad (2.10)$$

In both cases, we will conclude that the essential oscillation of u within smaller cylinder, centered at the origin, decreases in a way that can be quantitatively measured.

The analysis of this alternative leads to the main result of this work.

Proposition 2.6. *There exist constants $\eta \in (0, 1)$ and C , $\lambda > 1$, that can be determined only on terms of the data, satisfying the following. For $n=0,1,2,\dots$ construct the sequences*

$$R_n = C^{-n} R, \quad \omega_n = \eta^n \omega,$$

and the cylinders

$$Q_n = Q(R_n^{p^+}, c_n R_n) \text{ with } c_n = \left(\frac{\omega_n}{2^\lambda} \right)^{\frac{p^- - 2}{p^+}}.$$

Then, for all $n=0,1,2,\dots$,

$$Q_{n+1} \subset Q_n \text{ and } \operatorname{ess\,osc}_{Q_n} u \leq \omega_n.$$

As an immediate consequence, we obtain the following.

Theorem 2.7. *Under assumptions (1.2)-(1.5), any locally bounded weak solution of (1.1) is locally Hölder continuous.*

Remark 2.8. *By using the result in Proposition 2.6, the proof of Theorem 2.7 follows from a slight modification of the argument in Theorem 1.1 in [3].*

3 Local energy and logarithmic estimates

Let τ and ρ be so small that $Q(\tau, \rho) \subset \Omega_T$. Let ξ denote a piecewise smooth cutoff function in $Q(\tau, \rho)$ such that

$$\xi \in [0, 1], \quad |\nabla \xi| < \infty \text{ and } \xi(x, t) = 0 \text{ for } x \text{ outside } K_\rho.$$

Proposition 3.1. *Let u be a local weak solution of (1.1) in Ω_T . There exist positive constants C and C' such that, for every cylinder $Q(\tau, \rho) \subset \Omega_T$ and for every $k \in \mathbb{R}$*

$$\begin{aligned} & \sup_{-\tau < t < 0} \int_{K_\rho} (u - k)_\pm^2 \xi^{p^+}(x, t) dx + C \int_{-\tau}^t \int_{K_\rho} |\nabla(u - k)_\pm|^{p^-} \xi^{p^+} dx dt \\ & \leq \int_{K_\rho} (u - k)_\pm^2 \xi^{p^+}(x, -\tau) dx + C' \left[\int_{-\tau}^t \int_{K_\rho} (u - k)_\pm^2 \xi^{p^+ - 1} \xi_t dx dt \right. \\ & \quad + \int_{-\tau}^t \int_{K_\rho} (u - k)_\pm^{p^+} \left(|\nabla \xi|^{p^+} + \xi^{p^+} \right) dx dt \\ & \quad \left. + \int_{-\tau}^t \int_{K_\rho} \chi((u - k)_\pm > 0) dx dt \right]. \end{aligned} \tag{3.1}$$

Proof. See Proposition 3.1 in [7] □

Now, introduce the logarithmic function

$$\psi^\pm(u) = \psi(H_k^\pm, (u - k)_\pm, c) = \left(\ln \left(\frac{H_k^\pm}{H_k^\pm - (u - k)_\pm + c} \right) \right)_+,$$

where $H_k^\pm = \operatorname{ess\,sup}_{Q(\tau, \rho)} |(u - k)_\pm|$ and $0 < c < H_k^\pm$. In the cylinder $Q(\tau, \rho)$, we take a cutoff function satisfying $\xi \in [0, 1]$, $|\nabla \xi| < \infty$ and ξ is independent of $t \in (-\tau, 0)$.

Proposition 3.2. *Let u be local weak solution of (1.1) in Ω_T . There exists a positive constant C such that for every cylinder $Q(\tau, \rho) \in \Omega_T$ and for every*

level $k \in \mathbb{R}$,

$$\begin{aligned}
& \operatorname{ess\,sup}_{-\tau < t < 0} \int_{K_\rho} [\psi^\pm(u)]^2 \xi^{p^+} dx \\
& \leq \int_{K_\rho \times \{-\tau\}} [\psi^\pm(u)]^2 \xi^{p^+} dx + C \left(\int_{-\tau}^0 \int_{K_\rho} \psi^\pm(u) [(\psi^\pm)'(u)]^2 \xi^{p^+} dx dt \right. \\
& \quad + \int_{-\tau}^0 \int_{K_\rho} \psi^\pm(u) [(\psi^\pm)'(u)]^{2-p^-} \left(|\nabla u|^{p^+} + 1 + \xi^{p^+} \right) dx dt \\
& \quad + \int_{-\tau}^0 \int_{K_\rho} \psi^\pm(u) \left(|\nabla u|^{p^+} + 1 + \xi^{p^+} \right) dx dt \\
& \quad \left. + \int_{-\tau}^0 \int_{K_\rho} |u|^{p^+} \psi^\pm(u) [(\psi^\pm)'(u)]^2 \xi^{p^+} dx dt \right). \tag{3.2}
\end{aligned}$$

Proof. See Proposition 3.2 in [7] □

4 Rescaled iteration

The following rescaled iteration technique applies to any subcylinder of Ω_T and it is crucial in both alternatives (2.9) and (2.10). Let $m > 0$ be given by $m = m_1 + m_2$, where $m_1 \geq 1$, and $m_2 \geq 0$ and consider the cube

$$K_{d_1 R} = \left\{ x \in \mathbb{R}^N, \max_{1 \leq i \leq N} |x_i| < d_1 R \right\}, \quad d_1 = \left(\frac{\omega}{2^{m_1}} \right)^{\frac{p^- - 2}{p^+}},$$

and the box

$$Q_R(m_1, m_2) = K_{d_1 R} \times \left(-2^{m_2(p^- - 2)} R^{p^+}, 0 \right).$$

Fix $(\bar{x}, \bar{t}) \in \Omega_T$, and let $R > 0$ be so small that

$$(\bar{x}, \bar{t}) + Q_R(m_1, m_2) \subset \Omega_T.$$

Lemma 4.1. *There exists a number ν_0 that can be determined a priori only in terms of the data, such that:*

(1) *If u is a super-solution of (1.1) in $(\bar{x}, \bar{t}) + Q_R(m_1, m_2)$ satisfying*

$$\operatorname{ess\,osc}_{(\bar{x}, \bar{t}) + Q_R(m_1, m_2)} u \leq \omega$$

and

$$\left| \left\{ (x, t) \in (\bar{x}, \bar{t}) + Q_R(m_1, m_2) : u(x, t) < \mu^- + \frac{\omega}{2^m} \right\} \right| \leq \nu_0 |Q_R(m_1, m_2)|,$$

then

$$u(x, t) \geq \mu^- + \frac{\omega}{2^{m+1}}, \quad \forall (x, t) \in (\bar{x}, \bar{t}) + Q_{\frac{R}{2}}(m_1, m_2).$$

(2) If u is a sub-solution of (1.1) in $(\bar{x}, \bar{t}) + Q_R(m_1, m_2)$ satisfying

$$\underset{(\bar{x}, \bar{t}) + Q_R(m_1, m_2)}{\text{ess osc}} u \leq \omega$$

and

$$\left| \left\{ (x, t) \in (\bar{x}, \bar{t}) + Q_R(m_1, m_2) : u(x, t) > \mu^+ - \frac{\omega}{2^m} \right\} \right| \leq \nu_0 |Q_R(m_1, m_2)|,$$

then

$$u(x, t) \leq \mu^+ - \frac{\omega}{2^{m+1}}, \quad \forall (x, t) \in (\bar{x}, \bar{t}) + Q_{\frac{R}{2}}(m_1, m_2).$$

Proof. We only prove the statement regarding super-solutions (for sub-solutions the proof is similar). Assume $(\bar{x}, \bar{t}) = (0, 0)$ and construct the decreasing sequences of numbers

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \quad k_n = \mu^- + \frac{\omega}{2^{m+1}} + \frac{\omega}{2^{m+n+1}}, \quad n = 0, 1, 2, \dots,$$

and the families of nested cubes and cylinders

$$K_n = K_{d_1 R_n}, \quad d_1 = \left(\frac{\omega}{2^{m_1}} \right)^{\frac{p^- - 2}{p^+}},$$

$$Q_n = Q_{R_n}(m_1, m_2) = K_n \times \left(-2^{(p^- - 2)m_2} R_n^{p^+}, 0 \right).$$

Let $0 < \xi_n(x, t) \leq 1$ be piecewise smooth functions in Q_n such that

$$\begin{cases} \xi_n = 1 \text{ in } Q_{n+1}, \quad \xi_n = 0 \text{ on } \partial Q_n, \\ |\nabla \xi_n| \leq \frac{2^{n+2}}{R} \left(\frac{\omega}{2^{m_1}} \right)^{\frac{2-p^-}{p^+}}, \text{ and } 0 \leq \xi_{n,t} \leq 2^{(2-p^-)m_2} \times \frac{2^{p^+(n+2)}}{R^{p^+}}. \end{cases}$$

Now, by using the energy inequality (3.1) for the function $(u - k_n)_-$ we get

$$\begin{aligned}
& \sup_{-2^{(p^- - 2)m_2} R_n^{p^+} < t < 0} \int_{K_n} (u - k_n)_-^2 \xi_n^{p^+}(x, t) dx \\
& + \int \int_{Q_n} |\nabla(u - k_n)_-|^{p^-} \xi_n^{p^+} dx dt \\
& \leq C \left[\frac{2^{(2-p^-)m_2} 2^{p^+(n+2)}}{R^{p^+}} \int \int_{Q_n} (u - k_n)_-^2 dx dt \quad (4.1) \right. \\
& + \frac{2^{(n+2)p^+}}{R^{p^+}} \left(\frac{\omega}{2^{m_1}} \right)^{2-p^-} \int \int_{Q_n} (u - k_n)_-^{p^+} dx dt \\
& \left. + \int \int_{Q_n} \chi_{((u-k_n)_- > 0)} dx dt \right].
\end{aligned}$$

Since the solutions are assumed to be bounded we can always normalize and assume, w.l.g., that

$$\omega \leq 1, \quad (4.2)$$

then

$$\frac{\omega}{2^{m_1}} < 1 \implies \left(\frac{\omega}{2^{m_1}} \right)^{2-p^-} \leq \left(\frac{\omega}{2^{m_1}} \right)^{2-p^+}.$$

Also, by using the fact that

$$(u - k_n)_- \leq \sup_{Q_n} (u - k_n)_- \leq \frac{\omega}{2^{m+1}} + \frac{\omega}{2^{m+1+n}} \leq \frac{\omega}{2^m},$$

$$\left(\frac{\omega}{2^{m_1}} \right)^{2-p^+} \left(\frac{\omega}{2^m} \right)^{p^+} = \left(\frac{\omega}{2^m} \right)^2 2^{(2-p^+)m_2},$$

and by means of assumption (2.6) and Remark 2.5, the terms on the right hand side of (4.1) are estimated above by

$$C 2^{(2-p^-)m_2} \frac{2^{np^+}}{R^{p^+}} \left(\frac{\omega}{2^m} \right)^2 \int \int_{Q_n} \chi_{((u-k_n)_- > 0)} dx dt.$$

Setting

$$\bar{k}_n = \frac{k_n + k_{n+1}}{2},$$

then for all $t \in \left(-2^{(2-p^-)m_2} R_n^{p^+}, 0 \right)$, we have

$$\begin{aligned}
\int_{K_n} (u - k_n)_-^2 \xi_n^{p^+}(x, t) dx & \geq \int_{K_n} (k_n - \bar{k}_n)^{2-p^-} (u - k_n)_-^{p^-} \xi_n^{p^+}(x, t) dx \\
& = \left(\frac{\omega}{2^m} \right)^{2-p^-} 2^{(n+3)(p^- - 2)} \int_{K_n} (u - \bar{k}_n)_-^{p^-} \xi_n^{p^+}(x, t) dx,
\end{aligned}$$

and since $k_n > \bar{k}_n$, we have

$$\int \int_{Q_n} |\nabla(u - k_n)_-|^{p^-} \xi_n^{p^+} dxdt \geq \int \int_{Q_n} |\nabla(u - \bar{k}_n)_-|^{p^-} \xi_n^{p^+} dxdt.$$

By combining all the previous estimates and dividing through by

$$\left(\frac{\omega}{2^m}\right)^{2-p^-} 2^{(n+3)(p^- - 2)},$$

we obtain

$$\begin{aligned} & \sup_{-2^{(p^- - 2)m_2} R_n^{p^+} < t < 0} \int_{K_n} (u - \bar{k}_n)_-^{p^-} \xi_n^{p^+} dx \\ & + \left(\frac{\omega}{2^m}\right)^{p^- - 2} 2^{(n+3)(2-p^-)} \int \int_{Q_n} |\nabla(u - \bar{k}_n)_-|^{p^-} \xi_n^{p^+} dxdt \quad (4.3) \\ & \leq C \frac{2^{2n}}{R^{p^+}} 2^{(2-p^-)m_2} \left(\frac{\omega}{2^m}\right)^{p^-} \int \int_{Q_n} \chi_{((u - k_n)_- > 0)} dxdt. \end{aligned}$$

Next consider the change of variables

$$y = d_1^{-1}x, \quad z = 2^{(2-p^-)m_2}t$$

which maps the cylinders Q_n into the cylinders $Q'_n = K_{R_n} \times (-R_n^{p^+}, 0)$, and define new functions

$$\hat{u}(y, z) = u(d_1 y, 2^{(p^- - 2)m_2} z), \quad \hat{\xi}_n = \xi_n(d_1 y, 2^{(p^- - 2)m_2} z),$$

and the sets

$$A_n(z) = \{y \in K_{R_n} : \hat{u}(y, z) < K_n\}, \quad \text{with } |A_n| = \int_{-R_n^{p^+}}^0 |A_n(z)| dz.$$

Since $1 < p^- \leq p(x) \leq p^+ < 2$ and $\omega < 1$, the coefficient

$$\left(\frac{\omega}{2^m}\right)^{p^- - 2} 2^{(n+3)(2-p^-) + m_2(p^- - 2)} = \omega^{p^- - 2} 2^{(2-p^-)(m_1 + n + 3)} > 1,$$

then (4.3) becomes

$$\begin{aligned} & \sup_{-R_n^{p^+} < t < 0} \int_{K_{R_n}} (\hat{u} - \bar{k}_n)_-^{p^-} \hat{\xi}_n^{p^+} dz + \int \int_{Q'_n} |\nabla(\hat{u} - \bar{k}_n)_-|^{p^-} \hat{\xi}_n^{p^+} dydz \quad (4.4) \\ & \leq C \frac{2^{2n}}{R^{p^+}} \left(\frac{\omega}{2^m}\right)^{p^-} \int \int_{Q'_n} \chi_{((\hat{u} - k_n)_- > 0)} dydz. \end{aligned}$$

By Hölder's inequality, Proposition 3.1 of chapter I in [10], and (4.4), we obtain

$$\begin{aligned}
& \int_{-R_n^{p^+}}^0 \int_{K_{R_n}} (\hat{u} - \bar{k}_n)_-^{p^-} \hat{\xi}_n^{p^+} dydz \\
& \leq \left\{ \int_{-R_n^{p^+}}^0 \int_{K_{R_n}} \left[(\hat{u} - \bar{k}_n)_- \hat{\xi}_n^{\frac{p^+}{p^-}} \right]^{\frac{p^-(N+p^-)}{N}} dydz \right\}^{\frac{N}{N+p^-}} \\
& \quad \times \left\{ \int_{-R_n^{p^+}}^0 \int_{K_{R_n}} \chi_{((\hat{u}-\bar{k}_n)_->0)}^{\frac{N+p^-}{p^-}} dydz \right\}^{\frac{p^-}{N+p^-}} \\
& = \left\{ \int_{-R_n^{p^+}}^0 \int_{K_{R_n}} \left[(\hat{u} - \bar{k}_n)_- \hat{\xi}_n^{\frac{p^+}{p^-}} \right]^{\frac{p^-(N+p^-)}{N}} dydz \right\}^{\frac{N}{N+p^-}} |A_n|^{\frac{p^-}{N+p^-}} \quad (4.5) \\
& \leq C \left[\sup_{-R_n^{p^+} < z < 0} \int_{K_{R_n}} (\hat{u} - \bar{k}_n)_-^{p^-} \hat{\xi}_n^{p^+} dy \right]^{\frac{p^-}{N+p^-}} \\
& \quad \times \left[\int_{-R_n^{p^+}}^0 \int_{K_{R_n}} |\nabla(\hat{u} - \bar{k}_n)_-|^{p^-} \hat{\xi}_n^{p^+} dydz \right. \\
& \quad \left. + \int_{-R_n^{p^+}}^0 \int_{K_{R_n}} (\hat{u} - \bar{k}_n)_-^{p^-} |\nabla \hat{\xi}_n|^{p^-} dydx \right]^{\frac{N}{N+p^-}} |A_n|^{\frac{p^-}{N+p^-}} \\
& \leq C \frac{2^{2n}}{R^{p^+}} \left(\frac{\omega}{2^m} \right)^{p^-} |A_n|^{1+\frac{p^-}{N+p^-}}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int_{-R_n^{p^+}}^0 \int_{K_{R_n}} (\hat{u} - \bar{k}_n)_-^{p^-} \hat{\xi}_n^{p^+} dydz & \geq \int_{-R_{n+1}^{p^+}}^0 \int_{K_{R_{n+1}}} (\hat{u} - \bar{k}_n)_-^{p^-} dydz \\
& \geq |\bar{k}_n - k_{n+1}|^{p^-} |A_{n+1}| \quad (4.6) \\
& = 2^{-(n+3)p^-} \left(\frac{\omega}{2^m} \right)^{p^-} |A_{n+1}|.
\end{aligned}$$

Combining (4.5) and (4.6), we get that

$$|A_{n+1}| \leq C 4^{np^+} |A_n|^{1+\frac{p^-}{N+p^-}}. \quad (4.7)$$

Next, by defining the numbers

$$Y_n = \frac{|A_n|}{|Q'_n|},$$

we obtain the following recursive relation

$$Y_{n+1} \leq C4^{np^+} Y_n^{1 + \frac{p^-}{N+p^-}}.$$

Therefore, Lemma 4.1 of Chapter I in [10] implies that if

$$Y_0 \leq C^{-\frac{N+p^-}{p^-}} 4^{-p^+ \left(\frac{p^-+N}{p^-}\right)^2} = \nu_0, \quad (4.8)$$

then

$$Y_n \longrightarrow 0. \quad (4.9)$$

However, (4.8) is nothing but the assumption (2.9). Hence, the result easily follows from (4.9). \square

5 continuity of the weak solutions

In this section we analyze the alternative and prove proposition 2.6. Assume that there exists a cylinder of the type $\left[(\bar{x}, 0) + Q(R^{p^+}, d_0 R)\right]$ making up the partition of $Q(R^{p^+}, c_0 R)$ for which (2.9) holds. Then by applying Lemma 4.1 with $m_1 = 1$ and $m_2 = 0$, we conclude that

$$u(x, t) \geq \mu^- + \frac{\omega}{4}, \quad \forall (x, t) \in (\bar{x}, 0) + Q\left(\left(\frac{R}{2}\right)^{p^+}, d_0 \frac{R}{2}\right). \quad (5.1)$$

We view the box $\left[(\bar{x}, 0) + Q\left(\left(\frac{R}{2}\right)^{p^+}, d_0 \frac{R}{2}\right)\right]$ as a block inside $Q(R^{p^+}, c_0 R)$.

Let $R(\omega)$ be the radius introduced in (2.8). The location of \bar{x} within the cube $K_{R(\omega)}$ is only known qualitatively. We will show that the positivity of (5.1) spreads over the full cube $K_{c_0 R}$, for all times

$$-\left(\frac{R}{8}\right)^{p^+} \leq t \leq 0.$$

In a precise way we will prove the following

Proposition 5.1. *Assume that (2.9) holds for some $\bar{x} \in K_{R(\omega)}$. There exists a positive number s_1 that can be determined a priori in terms of the data, such that*

$$u(x, t) \geq \mu^- + \frac{\omega}{2^{s_1+1}}, \quad \forall (x, t) \in Q\left(\left(\frac{R}{8}\right)^{p^+}, c_0 R\right). \quad (5.2)$$

To prove Proposition 5.1, we regard \bar{x} as the center of a large cube $\bar{x} + K_{8c_0R}$ which we may assume to be contained in $K_{R^{\frac{p^+}{2}}}$. Indeed if not we could have

$$16c_0R > R^{\frac{p^+}{2}} \implies \omega \leq 16^{\frac{p^+}{2-p^-}} 2^\lambda R^{\frac{p^+}{2}},$$

then there will be nothing to prove. We work within the box

$$(\bar{x}, 0) + Q\left(\left(\frac{R}{2}\right)^{p^+}, 8c_0R\right),$$

and show that the conclusion of Proposition 5.1 holds within the cylinder

$$(\bar{x}, 0) + Q\left(\left(\frac{R}{8}\right)^{p^+}, 2c_0R\right).$$

This contains $Q\left(\left(\frac{R}{8}\right)^{p^+}, c_0R\right)$, regardless of the location of \bar{x} in the cube $K_{R(\omega)}$.

We begin by considering an auxiliary function

$$v = (u - \mu^-) \frac{2}{\omega}, \quad (5.3)$$

and introducing the following change of variables

$$x \mapsto \frac{x - \bar{x}}{2c_0R}, \quad t \mapsto \left(\frac{8}{R}\right)^{p^+} t. \quad (5.4)$$

Thus, the cylinder $(\bar{x}, 0) + Q\left(\left(\frac{R}{2}\right)^{p^+}, 8c_0R\right)$ is mapped into $Q_4 = K_4 \times (-4^{p^+}, 0)$. Denoting again with x and t the new variables, the function v satisfies the PDE

$$\frac{\partial v}{\partial t} - \operatorname{div} \tilde{A}(x, t, v, \nabla v) = \tilde{B}(x, t, v, \nabla v), \quad \text{in } D'(Q_4), \quad (5.5)$$

with

$$\tilde{A}(x, t, v, \nabla v) = \left(\frac{8}{R}\right)^{-p^+} \left(\frac{\omega}{2}\right)^{-1} \frac{1}{2c_0R} A(x, t, u, \nabla u), \quad (5.6)$$

and

$$\tilde{B}(x, t, v, \nabla v) = \left(\frac{8}{R}\right)^{-p^+} \left(\frac{\omega}{2}\right)^{-1} B(x, t, u, \nabla u). \quad (5.7)$$

By using (1.2)-(1.4) and assumptions (2.6) and (4.2), the applications $\tilde{A} : Q_4 \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$ and $\tilde{B} : Q_4 \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$ satisfies the following structure conditions

$$|\tilde{A}(x, t, v, \nabla v)| \leq C \left(\tilde{\phi}(x, t) + |v|^{p(x,t)-1} + |\nabla v|^{p(x,t)-1} \right), \quad (5.8)$$

$$|\tilde{B}(x, t, v, \nabla v)| \leq C \left(\tilde{\phi}(x, t) + |v|^{p(x,t)-1} + |\nabla v|^{p(x,t)-1} \right), \quad (5.9)$$

$$\tilde{A}(x, t, v, \nabla v) \cdot \nabla v \geq C |\nabla v|^{p(x,t)}, \quad (5.10)$$

where $\tilde{\phi}(x, t) \in L^\infty(\Omega_T)$ and C is a positive constant depend on p^+ , p^- , λ and R . Therefore, (5.1) becomes

$$v(x, t) \geq \frac{1}{2} \text{ a.e. } (x, t) \in Q(h_0) = \{x : |x| < h_0\} \times (-4^{p^+}, 0) \quad (5.11)$$

where

$$h_0 = \frac{d_0}{4c_0} = \frac{1}{4} \left(\frac{2}{2^\lambda} \right)^{\frac{2-p^-}{p^+}} = 2^{\frac{(\lambda-1)(p^- - 2)}{p^+} - 2} < 1. \quad (5.12)$$

We regard $Q(h_0)$ as a thin cylinder sitting at the center of Q_4 . We will prove that the relative largeness of v in $Q(h_0)$ spreads sidewise over Q_2 .

Proposition 5.1 will be a consequence of the following lemma.

Lemma 5.2. *For every $\nu \in (0, 1)$ there exists a positive number $\delta^* \in (0, 1)$, that can be determined a priori only in terms of ν , N , p^+ , p^- and the data, such that*

$$|\{x \in K_2 : v(x, t) \leq \delta^*\}| \leq \nu |K_2|, \quad (5.13)$$

for all time levels $t \in [-2p^+, 0]$.

Proof. The weak formulation of (5.5) is

$$\int_{K_4} v_t \varphi \, dx + \int_{K_4} \tilde{A}(x, t, v, \nabla v) \cdot \nabla \varphi \, dx = \int_{K_4} \tilde{B}(x, t, v, \nabla v) \varphi \, dx, \quad (5.14)$$

for all $-4^{p^+} < t < 0$ and all testing functions

$$\varphi \in W^{1,2}(-4^{p^+}, 0; L^2(K_4)) \cap L^{p(x,t)}(-4^{p^+}, 0; W_0^{1,p(x,t)}(K_4)).$$

Taking into account the structure of the function v and using Lemma 2.4, we can claim that the truncated function $(k - v)_+$ for all positive testing function φ , satisfies the following

$$\begin{aligned} \int_{K_4} \frac{\partial}{\partial \tau} (k - v)_+ \varphi(x, t) \, dx + \int_{K_4} \tilde{A}(x, t, v, \nabla((k - v)_+)) \cdot \nabla \varphi \, dx \\ \leq \int_{K_4} \tilde{B}(x, t, v, \nabla((k - v)_+)) \varphi \, dx, \end{aligned} \quad (5.15)$$

In this formulation we take the testing function

$$\varphi = \frac{\xi^{p^+}}{[k - (k - v)_+ + \delta k + R]^{p^- - 1}}, \quad (5.16)$$

where $\xi = \xi_1(x)\xi_2(t)$ is a piecewise smooth cutoff function in Q_4 satisfying

$$\begin{cases} 0 \leq \xi \leq 1 \text{ in } Q_4, \text{ and } \xi = 1 \text{ in } Q_2, \\ \xi = 0 \text{ on the parabolic boundary of } Q_4, \\ |\nabla \xi_1| \leq 1, \quad 0 \leq \xi_{2,t} \leq 1, \\ \text{the set } \{x \in K_4 \mid \xi_1(x) > k\} \text{ are convex } \forall k \in (0, 1). \end{cases} \quad (5.17)$$

Set also

$$\begin{cases} \phi_k(v) = \int_0^{(k-v)_+} \frac{ds}{[(1+\delta)k - s + R]^{p^- - 1}} \\ \psi_k(v) = \ln \left[\frac{k(1+\delta) + R}{(1+\delta)k - (k-v)_+ + R} \right]. \end{cases} \quad (5.18)$$

Here $k \in (0, \frac{1}{8})$, $\delta \in (0, \frac{1}{8})$ and $R \leq k\delta^{\frac{p^-}{p^- - 1}}$. Then, we obtain

$$\begin{aligned} & \int_{K_4} \frac{\partial}{\partial t} (k-v)_+ \frac{\xi^{p^+}}{[k(1+\delta) - (k-v)_+ + R]^{p^- - 1}} dx \\ & + \int_{K_4} \tilde{A}(x, t, v, \nabla(k-v)_+) \cdot \left[\frac{(p^- - 1)\nabla(k-v)_+ \xi^{p^+}}{[k(1+\delta) - (k-v)_+ + R]^{p^-}} \right. \\ & \left. + \frac{p^+ \xi^{p^+ - 1} \nabla \xi}{[k(1+\delta) - (k-v)_+ + R]^{p^- - 1}} \right] dx \\ & \leq \int_{K_4} \tilde{B}(x, t, v, \nabla(k-v)_+) \frac{\xi^{p^+}}{[k(1+\delta) - (k-v)_+ + R]^{p^- - 1}} dx. \end{aligned} \quad (5.19)$$

Since u is bounded, by simple computation we get that

$$\begin{aligned} 0 \leq C_1(p^+, p^-, \delta, k) & \leq [k(1+\delta) - (k-v)_+ + R] \\ & \leq C_2(p^+, p^-, \delta, k) < \infty. \end{aligned} \quad (5.20)$$

Now, estimating the various terms separately. Then by using (5.8)-(5.10), (5.20) and $\tilde{\phi}(x, t) \in L^\infty(\Omega_T)$, we obtain the following

$$\begin{aligned} \int_{K_4} \frac{\partial}{\partial t} \phi_k(v) \xi^{p^+} dx & = - \int_{K_4} \phi_k(v) p^+ \xi^{p^+ - 1} \xi_t dx \\ & + \frac{\partial}{\partial t} \int_{K_4} \phi_k(v) \xi^{p^+} dx, \end{aligned} \quad (5.21)$$

$$\begin{aligned}
& \int_{K_4} \frac{\tilde{A}(x, t, v, \nabla(k-v)_+) \nabla(k-v)_+ \xi^{p^+}}{[k(1+\delta) - (k-v)_+ + R]^{p^-}} dx \\
& \geq C \int_{K_4} \frac{|\nabla(k-v)_+|^{p(x,t)} \xi^{p^+}}{[k(1+\delta) - (k-v)_+ + R]^{p^-}} dx \\
& = C \int_{K_4} \frac{|\nabla(k-v)_+|^{p(x,t)} \xi^{p^+}}{[k(1+\delta) - (k-v)_+ + R]^{p(x,t)}} \\
& \quad \times [k(1+\delta) - (k-v)_+ + R]^{p(x,t)-p^-} dx \\
& \geq C \int_{K_4} |\nabla\psi_k(v)|^{p(x,t)} \xi^{p^+} dx,
\end{aligned} \tag{5.22}$$

also by using Young's inequality, and the fact that $0 \leq \xi \leq 1$ and $\frac{p(x,t)}{p(x,t)-1} \geq \frac{p^+}{p^+-1}$ imply that $\xi^{\frac{p(x,t)(p^+-1)}{p(x,t)-1}} \leq \xi^{p^+}$, we get that

$$\begin{aligned}
& \int_{K_4} \frac{\tilde{A}(x, t, v, \nabla(k-v)_+) \xi^{p^+-1} |\nabla\xi|}{[k(1+\delta) - (k-v)_+ + R]^{p^--1}} dx \\
& \leq C \int_{K_4} \frac{\left(\tilde{\phi}(x, t) + |(k-v)_+|^{p(x,t)-1} + |\nabla(k-v)_+|^{p(x,t)-1} \right)}{[k(1+\delta) - (k-v)_+ + R]^{p^--1}} \\
& \quad \times \xi^{p^+-1} |\nabla\xi| dx \\
& \leq C \left(\int_{K_4} \frac{\xi^{p^+-1} |\nabla\xi|}{[k(1+\delta) - (k-v)_+ + R]^{p^--1}} dx \right. \\
& \quad \left. + \int_{K_4} |\nabla\psi_k(v)|^{p(x,t)-1} \xi^{p^+-1} |\nabla\xi| dx \right) \\
& \leq C \left(\varepsilon \int_{K_4} |\nabla\psi_k(v)|^{p(x,t)} \xi^{p^+} dx + C(\varepsilon) \int_{K_4} |\nabla\xi|^{p(x,t)} dx \right. \\
& \quad \left. + \int_{K_4} \frac{\xi^{p^+-1} |\nabla\xi|}{[k(1+\delta) - (k-v)_+ + R]^{p^--1}} dx \right).
\end{aligned} \tag{5.23}$$

By the same method, the term on the right hand side of (5.15) becomes

$$\begin{aligned} & \int_{K_4} \tilde{B}(x, t, v, \nabla(k-v)_+) \frac{\xi^{p^+}}{[k(1+\delta) - (k-v)_+ + R]^{p^- - 1}} dx \\ & \leq C \left(\varepsilon \int_{K_4} |\nabla \psi_k(v)|^{p(x,t)} \xi^{p^+} dx + C(\varepsilon) \int_{K_4} \xi^{p^+} dx \right. \\ & \quad \left. + \int_{K_4} \frac{\xi^{p^+}}{[k(1+\delta) - (k-v)_+ + R]^{p^- - 1}} dx \right). \end{aligned} \quad (5.24)$$

Next, by the previous definition of k , the integral involving ξ_t is majorised by $\frac{C}{(2-p^-)}$. Therefor, by combining all the previous estimates in (5.15) and using (5.20), we get

$$\frac{\partial}{\partial t} \int_{K_4} \phi_k(v) \xi^{p^+} dx + C \int_{K_4} |\nabla \psi_k(v)|^{p(x,t)} \xi^{p^+} dx \leq \frac{C}{2-p^-}.$$

Since $|\nabla \psi_k(v)|^{p^-} \leq |\nabla \psi_k(v)|^{p(x,t)} + 1$, and using Poincaré's inequality, we obtain

$$\frac{\partial}{\partial t} \int_{K_4} \phi_k(v) \xi^{p^+} dx + C \int_{K_4} |\psi_k(v)|^{p^-} \xi^{p^+} dx \leq \frac{C}{2-p^-}. \quad (5.25)$$

Introduce the quantities

$$Y_n = \sup_{-4p^+ \leq t \leq 0} \int_{K_4 \cap [v(\cdot, t) < \delta^n]} \xi^{p^+}(x, t) dx, \quad n = 0, 1, 2, \dots \quad (5.26)$$

Then by using (5.25), Lemma 6.2 in [3] (see also Proposition 7.1 of chapter IV in [10]) gives that for any $\nu \in (0, 1)$ being fixed, there exists a constant $\delta \in (0, \frac{1}{8})$ depending only on $N, p^+, p^-, \lambda, \nu$ and such that for all integer $n = 1, 2, \dots, i^*$ such that

$$i^* \leq \ln R^{-1} \ln^{-1} \delta^{-1} - \frac{p^-}{p^- - 1}, \quad (5.27)$$

we have either

$$Y_n \leq \nu, \quad (5.28)$$

or

$$Y_{n+1} \leq \max\{\nu, (1-\delta)Y_n\}. \quad (5.29)$$

Consequently, iterating (5.27)-(5.28) gives Iterating relations (5.25) and (5.26) under the condition (5.24) and using the inequality

$$Y_1 \leq |K_4| = 2^N |K_2|, \quad (5.30)$$

we obtain

$$Y_n \leq \max\{\nu, (1-\delta)^{n-1}Y_1\} \leq \max\{\nu, 2^N(1-\delta)^{i-1}|K_2|\}, \quad i = 2, \dots, i^*. \quad (5.31)$$

In (5.31), we choose the smallest n_0 for which $(1-\delta)^{n_0-1} \leq 2^{-N}\nu$ and set $\delta^{n_0} = \delta^*$. Hence for all $t \in [-2^{p^+}, 0]$, we come to the relation (5.13). \square

Now we are going to prove Proposition 5.1.

Proof. In Lemma 5.2 choose $\nu = \nu_0$ where ν_0 is the number claimed by Lemma 4.1, and determine $\delta^* = \delta^*(\nu_0)$. Let m_2 defined by

$$2^{-m_2} = \delta^*(\nu_0),$$

and apply Lemma 4.1 with $\mu^- = 0$, $\omega = 1$, $m_1 = 0$, and $R = 2$ over the boxes

$$(0, \bar{t}) + K_2 \times \left(-2^{m_2(p^- - 2)}2^{p^+}, 0\right) = (0, \bar{t}) + Q_2(0, m_2) \quad (5.32)$$

as long as they are contained in Q_2 , i.e., for \bar{t} satisfying

$$2^{m_2(p^- - 2)}2^{p^+} - 2^{p^+} \leq \bar{t} \leq 0. \quad (5.33)$$

Since (5.13) holds true for all time levels in $t \in [-2^+, 0]$, each such box satisfies

$$|\{(x, t) \in (0, \bar{t}) + Q_2(0, m_2) \mid v(x, t) \leq 2^{-m_2}\}| \leq \nu_0 |Q_2(0, m_2)|. \quad (5.34)$$

Therefore, by Lemma 4.1

$$v \geq 2^{-(m_2+1)}, \quad \forall (x, t) \in (0, \bar{t}) + Q_2(0, m_2), \quad (5.35)$$

for all $\bar{t} \in \left(2^{m_2(p^- - 2)}2^{p^+} - 2^{m_2(p^- - 2)}, 0\right)$. Since

$$\left(2^{m_2(p^- - 2)}2^{p^+} - 2^{p^+} - 2^{m_2(p^- - 2)}, 0\right) \supset (-1, 0).$$

We conclude that

$$v(x, t) \geq 2^{-(m_2+1)}, \quad \forall (x, t) \in Q_1.$$

Consequently, we obtain

$$u \geq \mu^- + \frac{\omega}{2^{m_2+2}}, \quad \forall (x, t) \in (\bar{x}, 0) + Q\left(\left(\frac{R}{8}\right)^{p^+}, 2c_0R\right), \quad (5.36)$$

and the Proposition 5.1 follows with $s_1 = m_2+1$, and $m_2 = -\log_2(\delta^*(\nu_0))$. \square

As an immediate consequence we get the reduction of the oscillation of u .

Corollary 5.3. *There exists a constant $\sigma_0 \in (0, 1)$ depending only on the data and λ , such that if (2.9) holds then*

$$\operatorname{ess\,osc}_{Q(\rho^{p^+}, c_0\rho)} u \leq \sigma_0 \omega, \quad \forall \rho \in \left(0, \frac{R}{8}\right). \quad (5.37)$$

Proof. The proof follows since $Q(\rho^{p^+}, c_0\rho) \subset Q\left(\left(\frac{R}{8}\right)^{p^+}, c_0R\right)$, where we take $\sigma_0 = 1 - \frac{1}{2^{s_1+1}}$. \square

Assume now that (2.10) holds for all cylinders $(\bar{x}, 0) + Q(R^{p^+}, d_0R)$ making up the partition of $Q(R^{p^+}, c_0R)$. Since

$$\mu^+ - \frac{\omega}{2} = \mu^- + \frac{\omega}{2},$$

we can rephrase (2.10) as

$$\begin{aligned} & \left| \left\{ (x, t) \in (\bar{x}, 0) + Q(R^{p^+}, d_0R) : u(x, t) > \mu^+ - \frac{\omega}{2} \right\} \right| \\ & \leq (1 - \nu_0) \left| Q(R^{p^+}, d_0R) \right|. \end{aligned} \quad (5.38)$$

Let n be a positive number to be chosen and arrange that $2^{n\frac{2-p^-}{p^+}}$ is an integer. Then we combine $2^{n\frac{2-p^-}{p^+}}$ of these cylinders to form boxes congruent to

$$Q(R^{p^+}, d_*R) = K_{d_*R} \times \left(-R^{p^+}, 0\right), \quad d_* = \left(\frac{\omega}{2^{n+1}}\right)^{\frac{p^- - 2}{p^+}} = d_0 2^{n\frac{2-p^-}{p^+}}. \quad (5.39)$$

Next, we consider cylinders of type $(\bar{x}, 0) + Q(R^{p^+}, d_*R)$. These are contained in $Q(R^{p^+}, c_0R)$ if the abscissa \bar{x} of their vertices ranges over the cube $K_{R'(\omega)}$, where

$$\begin{aligned} R'(\omega) &= \left\{ 2^{\lambda(2-p^-)} - 2^{(n+1)\frac{2-p^-}{p^+}} \right\} \omega^{\frac{p^- - 2}{p^+}} R \\ &= L_1 d_* R, \end{aligned}$$

where $L_1 = \left(2^{\lambda - (n+1)\frac{2-p^-}{p^+}} - 1\right)$. We will take $\lambda > n + 1$ and arrange that L_1 is an integer. Then, we regard $Q(R^{p^+}, c_0R)$ as the union, up to a set of measure

zero of L_1^N pairwise disjoint boxes each congruent to $Q(R^{p^+}, d_*R)$. Since each box $(\bar{x}, 0) + Q(R^{p^+}, d_*R)$ is the pairwise disjoint union of boxes $(\bar{x}, 0) + Q(R^{p^+}, d_0R)$, each of them satisfying (5.38). Therefore, we can rephrase (2.10) this time as

$$\left| \left\{ (x, t) \in (\bar{x}, 0) + Q(R^{p^+}, d_*R) : u(x, t) > \mu^+ - \frac{\omega}{2} \right\} \right| \leq (1 - \nu_0) \left| Q(R^{p^+}, d_*R) \right|, \quad (5.40)$$

for all cylinders $(\bar{x}, 0) + Q(R^{p^+}, d_*R)$ making up the partition of $Q(R^{p^+}, c_0R)$.

Lemma 5.4. *Let $(\bar{x}, 0) + Q(R^{p^+}, d_*R)$ be any box contained in $Q(R^{p^+}, c_0R)$ and satisfying (5.40). Then, there exists a time level*

$$t^* \in \left(-R^{p^+}, -\frac{\nu_0}{2} R^{p^+} \right),$$

such that, for all $s \geq 2$,

$$\left| \left\{ x \in \bar{x} + K_{d_*R}, u(x, t^*) > \mu^+ - \frac{\omega}{2^s} \right\} \right| \leq \left(\frac{1 - \nu_0}{1 - \frac{\nu_0}{2}} \right) |K_{d_*R}|. \quad (5.41)$$

Proof. In fact, if (5.41) doesn't hold, then (5.40) also doesn't hold. \square

The next lemma asserts that a property similar to (5.41) still holds for all time levels from t^* up to 0.

Lemma 5.5. *There exists a positive integer n such that for all $t^* < t < 0$*

$$\left| \left\{ x \in \bar{x} + K_{d_*R} : u(x, t) > \mu^+ - \frac{\omega}{2^{n+1}} \right\} \right| \leq \left(1 - \left(\frac{\nu_0}{2} \right)^2 \right) |K_{d_*R}|. \quad (5.42)$$

Proof. By translation we may assume that $\bar{x} = 0$. Consider the cylinders $K_{d_*R} \times (t^*, 0)$ and the level $k = \mu^+ - \frac{\omega}{2}$. Define

$$H_k^+ = \operatorname{ess\,sup}_{K_{d_*R} \times (t^*, 0)} \left(u - \left(\mu^+ - \frac{\omega}{2} \right) \right)_+ \leq \frac{\omega}{2}.$$

Choose $n \in N$ big enough so that

$$0 < c = \frac{\omega}{2^{n+1}} < H_k^+.$$

Then the logarithmic function ψ^+ is well defined and satisfies the following

$$\psi^+ \leq n \ln 2, \text{ since } \frac{H_k^+}{H_k^+ - u + k + c} \leq \frac{\frac{\omega}{2}}{c} = 2^n,$$

$$0 \leq (\psi^+)' = \frac{1}{H_k^+ - u + k + c} \leq \frac{1}{c},$$

and

$$\left| (\psi^+)' \right|^{2-p^-} \leq \left(\frac{\omega}{2^{n+1}} \right)^{p^- - 2}.$$

In the logarithmic inequality (3.11) applied to the function $(u - k)_+$ let $0 < \xi(x) \leq 1$ be a smooth cutoff function defined in $K_{d_* R}$ such that for some $\pi \in (0, 1)$

$$\xi = 1, \text{ on the cube } K_{(1-\pi)d_* R}, \text{ and } |\nabla \xi| \leq (\pi d_* R)^{-1}.$$

With these choices, for all $t^* < t < 0$, we obtain

$$\begin{aligned} & \int_{K_{(1-\pi)d_* R} \times \{t\}} [\psi^+(u)]^2 dx \leq \int_{K_{d_* R} \times \{t^*\}} [\psi^+(u)]^2 dx \\ & + C \left(\int_{t^*}^0 \int_{K_{d_* R}} \psi^+(u) [(\psi^+)'(u)]^2 dx dt \right. \\ & + \left(\frac{1}{\pi d_* R} \right)^{p^+} \int_{t^*}^0 \int_{K_{d_* R}} \psi^+(u) [(\psi^+)'(u)]^{2-p^-} dx dt \\ & + \left(\frac{1}{\pi d_* R} \right)^{p^+} \int_{t^*}^0 \int_{K_{d_* R}} \psi^+(u) dx dt \\ & \left. + \int_{t^*}^0 \int_{K_{d_* R}} |u|^{p^+} \psi^+(u) [(\psi^+)'(u)]^{2-p^-} dx dt \right) \\ & \leq \int_{K_{d_* R} \times \{t^*\}} [\psi^+(u)]^2 dx \\ & + C \left(n \ln 2 \left(\frac{\omega}{2^{n+1}} \right)^{-2} R^{p^+} + n \ln 2 \left(\frac{1}{\pi R} \right)^{p^+} R^{p^+} \right. \\ & + n \ln 2 \left(\frac{1}{\pi R} \right)^{p^+} \left(\frac{\omega}{2^{n+1}} \right)^{2-p^-} R^{p^+} + n \ln 2 \left(\frac{\omega}{2^{n+1}} \right)^{-2} R^{p^+} \Big) |K_{d_* R}| \\ & \leq n^2 (\ln 2)^2 \left(\frac{1 - \nu_0}{1 - \frac{\nu_0}{2}} \right) |K_{d_* R}| + \frac{Cn}{\pi^{p^+}} |K_{d_* R}|, \end{aligned}$$

where we used Lemma 5.4 and the assumptions (2.6) and (4.2).

The left hand side is estimated below by integrating over the smaller set

$$\left\{ x \in K_{(1-\pi)d_*R} : u(x, t) > \mu^+ - \frac{\omega}{2^{n+1}} \right\}.$$

On such a set, $\xi = 1$ and since ψ is a decreasing function of H_k^+ , we have

$$\psi^2 \geq \ln^2 \left(\frac{\frac{\omega}{2}}{\frac{\omega}{2^{1+n-1}}} \right) = (n-1)^2 \ln^2 2.$$

We carry this in the previous estimation and divide by $(n-1)^2 \ln^2 2$, to obtain for all $t^* < t < 0$

$$\begin{aligned} & \left| \left\{ x \in K_{(1-\pi)d_*R} : u(x, t) > \mu^+ - \frac{\omega}{2^{n+1}} \right\} \right| \\ & \leq \left[\left(\frac{n}{n-1} \right)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}} \right) + \frac{C}{n\pi^{p^+}} \right] |K_{d_*R}| \end{aligned} \quad (5.43)$$

On the other hand

$$\begin{aligned} & \left| \left\{ x \in K_{d_*R}, u(x, t) > \mu^+ - \frac{\omega}{2^{n+1}} \right\} \right| \\ & \leq \left| \left\{ x \in K_{(1-\pi)d_*R} : u(x, t) > \mu^+ - \frac{\omega}{2^{n+1}} \right\} \right| + N\pi |K_{d_*R}| \\ & \leq \left\{ \left(\frac{n}{n-1} \right)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}} \right) + \frac{C}{n\pi^{p^+}} + N\pi \right\} |K_{d_*R}|. \end{aligned} \quad (5.44)$$

The proof is complete once we choose π so small that $N\pi \leq \frac{3}{8}\nu_0^2$, then n so large that

$$\frac{C}{n\pi^{p^+}} \leq \frac{3}{8}\nu_0^2 \quad \text{and} \quad \left(\frac{n}{n-1} \right)^2 \leq \left(1 - \frac{\nu_0}{2} \right) (1 + \nu_0) > 1.$$

□

The information of Lemma 5.5 will be exploited to show that in a small cylinder about $(0, 0)$, the solution u is strictly bounded above by

$$\mu^+ - \frac{\omega}{2^m}, \quad \text{for some } m > n + 1.$$

The process also determines the number λ that defines the size of $Q(R^{p^+}, c_0R)$. To make this quantitative consider the box

$$Q\left(\beta R^{p^+}, c_0R\right), \quad \beta = \frac{\nu_0}{2}, \quad c_0 = \left(\frac{\omega}{2\lambda}\right)^{\frac{p^- - 2}{p^+}}.$$

We view $Q\left(\beta R^{p^+}, c_0 R\right)$ as partitioned into sub-boxes $(\bar{x}, 0) + Q\left(\beta R^{p^+}, d_* R\right)$ where \bar{x} takes finitely many points within the cube $K_{R'(\omega)}$. For each of these sub-cylinders Lemma 5.5 holds.

Lemma 5.6. *For every $\nu \in (0, 1)$ there exists a number m depending only on the data and independent of ω and R such that, for all cylinders $(\bar{x}, 0) + Q\left(\beta R^{p^+}, d_* R\right)$ making up the partition of $Q\left(\beta R^{p^+}, c_0 R\right)$, we have*

$$\left| \left\{ (x, t) \in (\bar{x}, 0) + Q\left(\beta R^{p^+}, d_* R\right) : u(x, t) > \mu^+ - \frac{\omega}{2^m} \right\} \right| \leq \nu \left| Q\left(\beta R^{p^+}, d_* R\right) \right|. \quad (5.45)$$

Proof. After a translation we may assume that $(\bar{x}, 0) = (0, 0)$. Set $s_0 = n + 1$ and consider the energy inequality (3.1) written for $(u - k)_+$,

$$k = \mu^+ - \frac{\omega}{2^s}, \quad s = s_0, s_0 + 1, s_0 + 2, \dots, m - 1,$$

over the cylinder $Q\left(\beta (2R)^{p^+}, 2d_* R\right)$. Over such a box

$$\left(u(x, t) - \left(\mu^+ - \frac{\omega}{2^s} \right) \right)_+ \leq \frac{\omega}{2^s} \quad \text{a.e. } (x, t) \in Q\left(\beta (2R)^{p^+}, 2d_* R\right).$$

The cutoff function $(x, t) \rightarrow \xi(x, t)$ is taken to satisfy

$$\begin{cases} \xi = 1, & \text{on } Q\left(\beta R^{p^+}, d_* R\right), \\ \xi = 0, & \text{on the parabolic boundary of } Q\left(\beta (2R)^{p^+}, 2d_* R\right), \\ |\nabla \xi| \leq \left(\frac{1}{d_* R} \right)^{\frac{p^-}{p^+}}, & 0 \leq \xi_t \leq \frac{2}{\nu_0 R^{p^-}}. \end{cases}$$

We put these estimates in (3.1) and discard the first non-negative term on the

left hand side. This gives

$$\begin{aligned}
& \int \int_{Q(\beta R^{p^+}, d_* R)} |\nabla(u-k)_+|^{p^-} dxdt \\
& \leq C \left[\frac{2}{\nu_0 R^{p^-}} \int \int_{Q(\beta(2R)^{p^+}, 2d_* R)} (u-k)_+^2 dxdt \right. \\
& \quad + \frac{1}{(d_* R)^{p^-}} \int \int_{Q(\beta(2R)^{p^+}, 2d_* R)} (u-k)_+^{p^+} dxdt \\
& \quad \left. + \int \int_{Q(\beta(2R)^{p^+}, 2d_* R)} \chi((u-k)_+ > 0) dxdt \right] \quad (5.46) \\
& \leq C \left[\frac{2}{R^{p^-}} \left(\frac{\omega}{2^s}\right)^2 + \left(\frac{1}{d_* R}\right)^{p^-} \left(\frac{\omega}{2^s}\right)^{p^+} \right. \\
& \quad \left. + 2^{n+p^+} \left(\frac{1}{d_* R}\right)^{p^-} \left(\frac{\omega}{2^s}\right)^{p^-} \left(\frac{1}{d_* R}\right)^{-p^-} \left(\frac{\omega}{2^s}\right)^{-p^+} \right] \\
& \quad \times |Q(\beta R^{p^+}, d_* R)|.
\end{aligned}$$

Since $\omega \leq 1$, then

$$\begin{aligned}
\left(\frac{\omega}{2^s}\right)^2 &= \left(\frac{\omega}{2^s}\right)^{2-p^-} \left(\frac{\omega}{2^s}\right)^{p^-} \leq \left(\frac{\omega}{2^{n+1}}\right)^{2-p^-} \left(\frac{\omega}{2^s}\right)^{p^-} \\
&= \left(\frac{\omega}{2^{n+1}}\right)^{\left(\frac{2-p^-}{p^+}\right)p^+} \left(\frac{\omega}{2^s}\right)^{p^-} \leq \left(\frac{1}{d_*}\right)^{p^-} \left(\frac{\omega}{2^s}\right)^{p^-},
\end{aligned}$$

and by means of the assumption (2.6) and Remark 2.5 we may estimate

$$\begin{aligned}
\left(\frac{\omega}{2^s}\right)^{-p^-} R^{p^-} d_*^{p^-} &\leq \left(\frac{\omega}{2^s}\right)^{-p^- + \frac{(p^- - 2)p^-}{p^+}} R^{p^-} \\
&\leq 1.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
& \int \int_{Q(\beta R^{p^+}, d_* R)} |\nabla(u-k)_+|^{p^-} dxdt \\
& \leq \frac{C}{(d_* R)^{p^-}} \left(\frac{\omega}{2^s}\right)^{p^-} |Q(\beta R^{p^+}, d_* R)|. \quad (5.47)
\end{aligned}$$

Now, we consider the levels

$$l = \mu^+ - \frac{\omega}{2^{s+1}}, \quad k = \mu^+ - \frac{\omega}{2^s}, \quad s \geq s_0.$$

By virtue of Lemma 5.5, we get

$$\left| \left\{ x \in K_{d_* R}, u(\cdot, t) < \mu^+ - \frac{\omega}{2^s} \right\} \right| \geq \left(\frac{\nu_0}{2} \right)^2 |K_{d_* R}|, \quad \forall t \in (-\beta R^{p^+}, 0). \quad (5.48)$$

Setting

$$A_s(t) = \left\{ x \in K_{d_* R}, u(\cdot, t) > \mu^+ - \frac{\omega}{2^s} \right\}, \quad A_s = \int_{-\beta R^{p^+}}^0 |A_s(t)| \, dt.$$

Then, from Lemma 2.2 of chap I in [10], we obtain

$$\left(\frac{\omega}{2^{s+1}} \right) |A_{s+1}(t)| \leq \frac{C}{\nu_0^2} d_* R \int_{A_s(t) \setminus A_{s+1}(t)} |\nabla u| \, dx, \quad \forall t \in (-\beta R^{p^+}, 0). \quad (5.49)$$

Let us integrate both sides of (5.49) over the interval $(-\beta R^{p^+}, 0)$. Then, by using Hölder's inequality and raising the resulting relation to the power p^- , we obtain

$$\begin{aligned} & \left(\frac{\omega}{2^{s+1}} \right)^{p^-} |A_{s+1}(t)|^{p^-} \\ & \leq C (d_* R)^{p^-} \left(\int \int_{A_s} |\nabla u|^{p^-} \, dx \right) |A_s \setminus A_{s+1}|^{p^- - 1} \\ & \leq C (d_* R)^{p^-} \frac{1}{(d_* R)^{p^-}} \left(\frac{\omega}{2^s} \right)^{p^-} \left| Q(\beta R^{p^+}, d_* R) \right| |A_s \setminus A_{s+1}|^{p^- - 1}. \end{aligned} \quad (5.50)$$

Then,

$$|A_{s+1}|^{\frac{p^-}{p^- - 1}} \leq C \left| Q(\beta R^{p^+}, d_* R) \right|^{\frac{1}{p^- - 1}} |A_s \setminus A_{s+1}|, \quad (5.51)$$

for all $s = s_0, s_0 + 1, \dots, m - 1$. Since $\mu^+ - \frac{\omega}{2^{s+1}} \leq \mu^+ - \frac{\omega}{2^m}$, the quantities $|A_{s+1}| \geq |A_m|$. We combine this fact to obtain

$$\sum_{s=s_0}^{m-1} |A_{s+1}|^{\frac{p^-}{p^- - 1}} \geq (m - s_0) |A_m|^{\frac{p^-}{p^- - 1}}.$$

Note also that $\sum_{s=s_0}^{m-1} |A_s \setminus A_{s+1}| \leq \left| Q(\beta R^{p^+}, d_* R) \right|$. Collecting these estimates, we arrive at

$$|A_m| \leq \frac{C}{(m - s_0)^{\frac{p^- - 1}{p^-}}} \left| Q(\beta R^{p^+}, d_* R) \right|,$$

and the proof is complete once we choose m sufficiently large so that

$$\left(\frac{C}{m - s_0} \right)^{\frac{p^- - 1}{p^-}} \leq \nu.$$

□

To proceed we return to the box $Q(\beta R^{p^+}, c_0 R)$ and recall that it is the finite union, up to a set of measure zero, of pairwise disjoint boxes $(\bar{x}, 0) + Q(\beta R^{p^+}, d_* R)$. Therefore Lemma 5.6 implies.

Corollary 5.7. *For every $\nu \in (0, 1)$, there exists a number m depending only upon the data and independent of ω and R such that*

$$\begin{aligned} & \left| \left\{ (x, t) \in Q(\beta R^{p^+}, c_0 R) : u(x, t) > \mu^+ - \frac{\omega}{2^m} \right\} \right| \\ & \leq \nu \left| Q(\beta R^{p^+}, c_0 R) \right|. \end{aligned} \quad (5.52)$$

We finally determine the size of the cylinder $Q(\beta R^{p^+}, c_0 R)$ and consequently the number λ . First, in Corollary 5.7 take $\nu = \nu_0$ and determine m accordingly. Then let m_2 be given by

$$\beta = \frac{\nu_0}{2} = 2^{m_2(p^- - 2)},$$

and assume that $m \geq m_2$. Determine λ from

$$\lambda = m_1, \text{ and } m = m_1 + m_2.$$

With these choices, the cylinder $Q(\beta R^{p^+}, c_0 R)$ coincides with the cylinder $Q(m_1, m_2)$ introduced in Lemma 4.1. By Corollary 5.7, we have

$$\left| \left\{ (x, t) \in Q_R(m_1, m_2) : u(x, t) > \mu^+ - \frac{\omega}{2^m} \right\} \right| \leq \nu_0 |Q_R(m_1, m_2)|,$$

Using Lemma 4.1 implies

$$u(x, t) \leq \mu^+ - \frac{\omega}{2^{m+1}}, \quad \forall (x, t) \in Q_{\frac{R}{2}}(m_1, m_2).$$

which leads us to the following statement.

Proposition 5.8. *Assume that (2.10) holds true for all cylinders $(\bar{x}, 0) + Q(R^{p^+}, d_0 R)$ making up the partition of $Q(R^{p^+}, c_0 R)$. Then, for all $0 < \rho < \frac{R}{2}$*

$$\operatorname{ess\,osc}_{Q(\beta \rho^{p^+}, c_0 \rho)} u \leq \sigma_1 \omega, \quad (5.53)$$

where $\sigma_1 = 1 - \frac{1}{2^{m+1}}$.

Now, we are going to prove our main result Proposition 2.6.

Proof. The proof of Proposition 2.6 follows by combining the two alternatives. The conclusion of the first alternative is that

$$\operatorname{ess\,osc}_{Q(\rho^{p^+}, c_0\rho)} u \leq \sigma_0\omega, \quad \forall \rho \in \left(0, \frac{R}{8}\right), \quad (5.54)$$

where $\sigma_0 = 1 - \frac{1}{2^{s_1+1}}$. The conclusion of the second alternative is that

$$\operatorname{ess\,osc}_{Q(\beta\rho^{p^+}, c_0\rho)} u \leq \sigma_1\omega, \quad \forall \rho \in \left(0, \frac{R}{2}\right), \quad (5.55)$$

where $\sigma_1 = 1 - \frac{1}{2^{m+1}}$. Set

$$\sigma = \max\{\sigma_0, \sigma_1\}, \quad \text{and} \quad C = \frac{\beta^{\frac{1}{p^+}}}{4} = \left(\frac{\nu_0}{2^{2p^++1}}\right)^{\frac{1}{p^+}}.$$

Observe that, assuming $\nu_0 \leq \frac{1}{2}$,

$$c \leq \frac{1}{2^{2+\frac{2}{p^+}}} < \frac{1}{8} < \frac{1}{2}, \quad \text{and} \quad C \leq \frac{\beta^{\frac{1}{p^+}}}{2}.$$

Define

$$R_1 = CR, \quad \omega_1 = \sigma\omega,$$

and the cylinder

$$Q_1 = Q\left(R_1^{p^+}, C_1R_1\right), \quad C_1 = \left(\frac{\omega_1}{2^\lambda}\right)^{\frac{p^- - 2}{p^+}}.$$

Since $\sigma < 1$,

$$C_1R_1 \leq c_0R_1.$$

Therefore, combining both alternatives,

$$\operatorname{ess\,osc}_{Q_1} u \leq \omega_1. \quad (5.56)$$

The process can now be repeated inductively starting from such relation. \square

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